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# NON-CONSERVATIVE STABILITY OF INTERMEDIATE SPRING SUPPORTED COLUMNS WITH AN ELASTICALLY RESTRAINED END SUPPORT

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**Abstract**—The non-conservative stability of an intermediate spring supported uniform column elastically restrained at one end and subjected to a follower force at the other unsupported end is studied. It is found that when the intermediate spring support is far from the unsupported end, the instability mechanism is flutter. As the intermediate spring support approaches the unsupported end, the instability mechanism is changed from flutter to divergence with the increase of intermediate spring stiffness. For the hinged-intermediate and guided-intermediate spring supported columns, the critical buckling load of flutter instability will first decrease, then increase as the intermediate spring stiffness is increased. Nevertheless, when the instability mechanism is divergence, the critical buckling load depends on the location of the intermediate spring support only, whereas for the clamped-intermediate spring supported column the critical buckling load of divergence instability decreases monotonically to a fixed value as the intermediate spring stiffness is increased. Finally, the influence of elastic end restraints on the stability of the column is also investigated.

## 1. INTRODUCTION

The stability of an elastic uniform column loaded by an end follower force has been considered in many references (Herrmann, 1967; Bolotin, 1979; Leipholtz, 1980). The loss of stability of such a system may be due to flutter or divergence types, depending on the nature of the boundary conditions. It is well known that if the type of instability is divergence, the buckling loads of the system can be determined by a static approach, while for flutter the buckling loads should be determined by using the dynamic criterion (Ziegler, 1977).

In studies of the influence of intermediate support on the instability of the system, Zorii and Chernukha (1971) investigated the influence of the location of intermediate hinged support on the stability of a column simply supported at one end and subjected to a follower force at the other unsupported end. Sugiyama *et al.* (1985) discussed the effect of an intermediate lateral spring support on the stability of a cantilevered tubular pipe conveying fluid. Recently, Elishakoff and Lottati (1988) examined the exact solution of the problem of an intermediate hinged supported column with simply supported or clamped boundary conditions. It was shown that the nature of buckling of the system changes as the support position changes. The closely related problem of the stability of elastic columns subjected to a follower force at one end has been studied by Sundararajan (1976), Kar (1980) and Kounadis (1981), and recently by Rosa and Franciosi (1990).

In this paper, we study the stability of an intermediate spring supported uniform column elastically restrained at one end and subjected to a follower force at the other unsupported end. The influence of elastic end restraints and the location and stiffness of the intermediate spring support on the stability of the column is investigated. Several new observations are presented. While taking the limit study, the results are compared with those in the existing literature.

## 2. ANALYSIS

Consider the small transverse oscillations $\bar{V}(X, t)$ in the $XY$-plane of an elastic uniform Bernoulli–Euler column with general elastic restraints at one end and a spring support at intermediate location $X = X_I$, subjected to a follower force $P$ at the unsupported end, as shown in Fig. 1. The governing equation is

$$EI\frac{\partial^4 \bar{V}}{\partial X^4} + P\frac{\partial^2 \bar{V}}{\partial X^2} + m\frac{\partial^2 \bar{V}}{\partial t^2} = 0, \quad X \in (0, l). \tag{1}$$

The associated boundary conditions are

$$\text{at} \quad X = 0: \; EI\frac{\partial^2 \bar{V}}{\partial X^2} - K_{\theta L}\frac{\partial \bar{V}}{\partial X} = 0$$
$$EI\frac{\partial^3 \bar{V}}{\partial X^3} + K_{TL}\bar{V} = 0 \tag{2}$$

and

$$\text{at} \quad X = l: \; EI\frac{\partial^2 \bar{V}}{\partial X^2} = 0$$
$$EI\frac{\partial^3 \bar{V}}{\partial X^3} = 0, \tag{3}$$

and the continuity conditions at the support, $X = X_I$, are

$$\bar{V}|_{X_I^-} = \bar{V}|_{X_I^+}, \quad \left.\frac{\partial \bar{V}}{\partial X}\right|_{X_I^-} = \left.\frac{\partial \bar{V}}{\partial X}\right|_{X_I^+},$$
$$EI\left.\frac{\partial^2 \bar{V}}{\partial X^2}\right|_{X_I^-} = EI\left.\frac{\partial^2 \bar{V}}{\partial X^2}\right|_{X_I^+},$$
$$EI\left.\frac{\partial^3 \bar{V}}{\partial X^3}\right|_{X_I^-} = EI\left.\frac{\partial^3 \bar{V}}{\partial X^3}\right|_{X_I^+} + K_{TI}\bar{V}|_{X_I^+} \tag{4}$$

where $E$, $I$, $m$ and $l$ denote the Young's modulus, the second moment of area, mass per unit length and length of the beam, respectively. $K_{\theta L}$ and $K_{TL}$ are rotational and translational spring stiffnesses at $X = 0$ and $K_{TI}$ is the intermediate translational spring stiffness at $X = X_I$.

Using the modal approach, one assumes

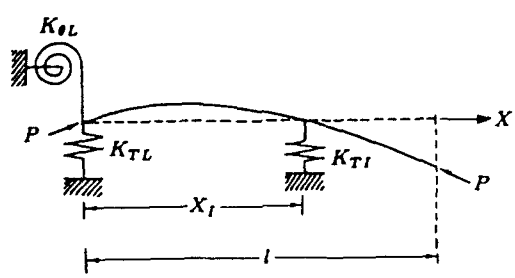


Fig. 1. Geometry and dimensions of the non-conservative system.

$$\bar{V}(X,t) = W(X)\,e^{i\omega t}, \tag{5}$$

where $i = \sqrt{-1}$ and $\omega$ is the natural frequency of vibration. Substituting eqn (5) into eqns (1)–(4), and using the following non-dimensional parameters,

$$x = \frac{X}{l},\quad \zeta = \frac{X_t}{l},\quad V = \frac{W}{l},\quad p = \frac{Pl^2}{EI},$$

$$\Lambda^2 = \frac{m\omega^2 l^4}{EI},\quad \beta_{\theta L} = \frac{K_{\theta L} l}{EI},\quad \beta_{TL} = \frac{K_{TL} l^3}{EI},\quad \beta_{Tt} = \frac{K_{Tt} l^3}{EI}, \tag{6}$$

we obtain the equation of motion in the following non-dimensional form

$$\frac{d^4V}{dx^4} + p\frac{d^2V}{dx^2} - \Lambda^2 V = 0, \tag{7}$$

with the associated boundary conditions

$$\text{at}\quad x = 0: \frac{d^2V}{dx^2} - \beta_{\theta L}\frac{dV}{dx} = 0$$

$$\frac{d^3V}{dx^3} + \beta_{TL} V = 0 \tag{8}$$

$$\text{at}\quad x = 1: \frac{d^2V}{dx^2} = 0$$

$$\frac{d^3V}{dx^3} = 0, \tag{9}$$

and the continuity conditions at $x = \zeta$,

$$V|_{\zeta^-} = V|_{\zeta^+},\quad \left.\frac{dV}{dx}\right|_{\zeta^-} = \left.\frac{dV}{dx}\right|_{\zeta^+},$$

$$\left.\frac{d^2V}{dx^2}\right|_{\zeta^-} = \left.\frac{d^2V}{dx^2}\right|_{\zeta^+},$$

$$\left.\frac{d^3V}{dx^3}\right|_{\zeta^-} = \left.\frac{d^3V}{dx^3}\right|_{\zeta^+} + \beta_{Tt} V|_{\zeta^+}. \tag{10}$$

The complete solution of eqn (7) is

$$V(x) = \begin{cases} C_1\cosh\lambda x + C_2\sinh\lambda x + C_3\cos\eta x + C_4\sin\eta x, & 0 < x < \zeta, \\ C_5\cosh\lambda x + C_6\sinh\lambda x + C_7\cos\eta x + C_8\sin\eta x, & \zeta < x < 1, \end{cases} \tag{11}$$

where

$$\lambda^2 = \frac{-p}{2} + \sqrt{\left[\left(\frac{p}{2}\right)^2 + \Lambda^2\right]},$$

$$\eta^2 = \frac{p}{2} + \sqrt{\left[\left(\frac{p}{2}\right)^2 + \Lambda^2\right]}, \tag{12}$$

and $C_i$, $i = 1, 2, \ldots, 8$ are constants to be determined. Substituting the solution (11) into the boundary conditions (8) and (9) and the continuity conditions (10) yields

$$[D_{ij}]\{C\} = \{0\}, \quad i, j = 1, 2, 3, 4, \tag{13}$$

where $\{C\}^T = [C_3, C_4, C_7, C_8]$ and $D_{ij}$ are given in the Appendix.

For a non-trivial solution of $V(x)$, the determinant

$$|D_{ij}| = 0. \tag{14}$$

This leads to the characteristic equation of the buckling loads. The lowest value of the buckling loads is the critical buckling load and the corresponding instability mechanism may be divergence or flutter.

*Divergence instability*

If the instability mechanism of the problem is divergence, the critical buckling load can be determined through the static method. By ignoring the inertial term in eqn (7), the complete solution of the system, eqn (11), becomes

$$V(x) = \begin{cases} C_1 + C_2 x + C_3 \cos \eta x + C_4 \sin \eta x, & 0 < x < \zeta, \\ C_5 + C_6 x + C_7 \cos \eta x + C_8 \sin \eta x, & \zeta < x < 1, \end{cases} \tag{15}$$

where

$$\eta^2 = p. \tag{16}$$

Now, two special cases are discussed.

*Case A. With translational and without rotational elastic end support*

For a column with translational and without rotational elastic support at $X = 0$, $\beta_{0L} = 0$. Consequently, the characteristic equation for divergence instability is

$$\sin \eta\zeta = 0. \tag{17}$$

As a result, the critical buckling load is

$$p_{cr} = \frac{\pi^2}{\zeta^2}. \tag{18}$$

It can be observed that the critical buckling load for divergence instability depends on the location of the intermediate support only. It is independent of the stiffness of the intermediate spring support and that of the translational elastic end support. Obviously, this is a generalization of the result given by Elishakoff and Lottati (1988). They had shown that the critical buckling load for the divergence instability of a simply supported column with intermediate hinged support depends on the location of the intermediate hinged support only.

*Case B. With rotational and without translational elastic end support*

When the column is subjected to rotational and without translational elastic end support at $X = 0$, then $\beta_{TL} = 0$. For divergence instability, the characteristic equation is

$$\cos \eta\zeta = 0. \tag{19}$$

Consequently, the critical buckling load is

$$p_{cr} = \frac{\pi^2}{4\zeta^2}. \quad (20)$$

This indicates that the critical buckling load for divergence instability also depends on the location of the intermediate support only.

## 3. NUMERICAL RESULTS AND DISCUSSION

Figure 2 illustrates the variation of critical buckling load ($p_{cr}$) versus the intermediate spring stiffness ($\beta_{TI}$) of a hinged-intermediate spring supported column for several locations of the intermediate spring support ($\zeta$). The curves for lower values of $\zeta$ (such as $\zeta = 0.1$, 0.3 and 0.4) show that the instability mechanism is flutter, which is due to the coalescence of the first and second natural frequencies, while those for higher values of $\zeta$ (such as $\zeta = 0.8$, 0.803 and 1.0) show that stability is lost owing to divergence and the critical buckling load depends on $\zeta$ only and decreases as $\zeta$ is increased. One observes that the critical buckling load of flutter instability will first decrease as $\beta_{TI}$ is increased and as $\beta_{TI}$ is greater than the corresponding critical value, which corresponds to the lowest critical buckling load, it will increase instead. For intermediate values of $\zeta$ ($\zeta = 0.5$, 0.52 and 0.6), the curves show that the instability mechanism changes from flutter to divergence at a transition point of $\beta_{TI}$. At this transition point there is a sudden increase in the value of critical buckling load and thereafter the divergence critical buckling load remains constant with increasing $\beta_{TI}$. The value of $\beta_{TI}$ of the transition point decreases with increasing $\zeta$.

Figure 3 illustrates the variation of critical buckling load versus the intermediate spring stiffness of a guided-intermediate spring supported column for several locations of the intermediate spring support. These curves are qualitatively similar to those in Fig. 2. However, comparison of Fig. 3 with Fig. 2 shows that for the same $\zeta$ the critical buckling load and the transition value of $\beta_{TI}$ of the guided-intermediate spring supported column are less than those of the hinged-intermediate spring supported column. It can also be found that the critical buckling load of divergence instability depends on the location of the intermediate spring support only. Especially for the curve for $\zeta = 0.3$, the critical buckling load of flutter instability due to the coalescence of the first and second natural frequencies decreases, firstly to the lowest critical buckling load, then increases with the increase of $\beta_{TI}$ to the value of jump where the instability mechanism changes from flutter to divergence. This is due to the buckling load of divergence instability being less than that of flutter

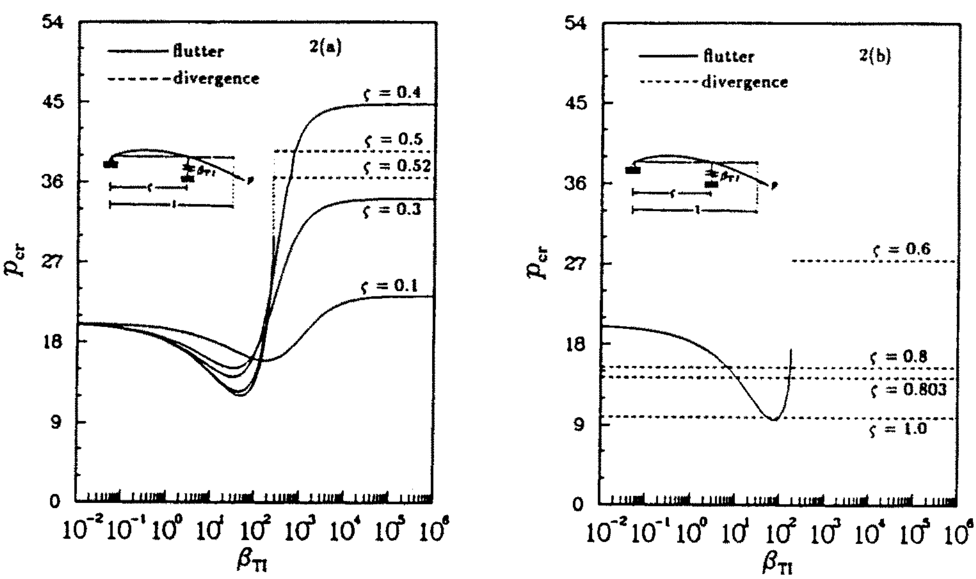


Fig. 2. Variation of the critical buckling load of a hinged-intermediate spring supported column with the intermediate spring support stiffness.

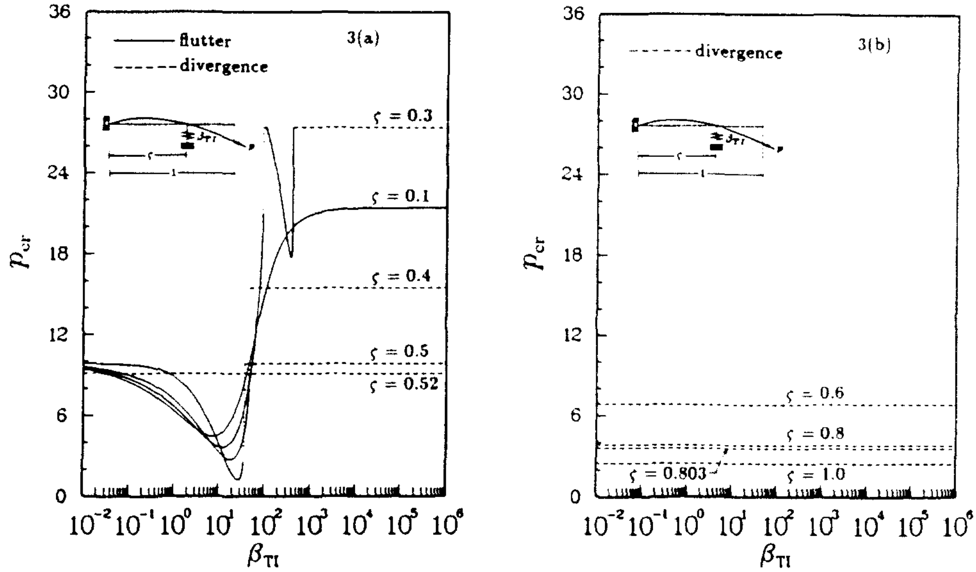


Fig. 3. Variation of the critical buckling load of a guided-intermediate spring supported column with the intermediate spring support stiffness.

instability, resulting from the coalescence of the second and third natural frequencies. This phenomenon exists for a small range of $\beta_{TI}$. Nevertheless, the buckling load of flutter instability decreases with the increase of $\beta_{TI}$, then the buckling load of flutter instability becomes less than that of divergence instability and the instability mechanism changes from divergence to flutter. However, there exists no jump of the critical buckling load. With a further increase of $\beta_{TI}$, the critical buckling load of flutter instability decreases firstly, then increases and finally the instability mechanism becomes divergence instability again.

Figure 4 illustrates the variation of critical buckling load versus the intermediate spring stiffness of a clamped-intermediate spring supported column for several locations of the intermediate spring support. The curves for lower values of $\zeta$ (such as $\zeta = 0.1$, 0.3 and 0.4) show that the instability mechanism is flutter, which is due to the coalescence of the first and second frequencies. For the curves of $\zeta = 0.52$ and $\zeta = 0.6$, the changes of critical buckling load and instability mechanism are similar to those for the curve of $\zeta = 0.3$ in Fig.

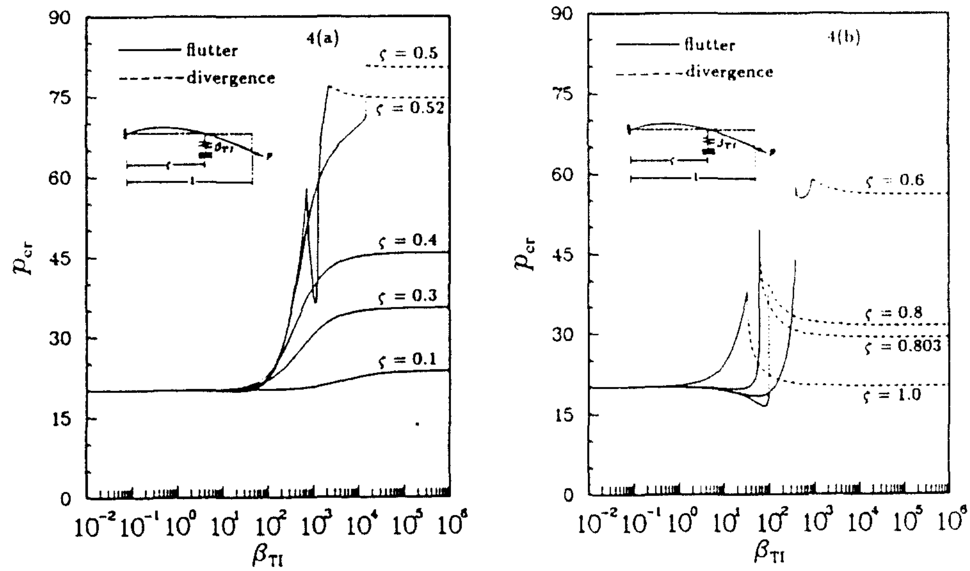


Fig. 4. Variation of the critical buckling load of a clamped-intermediate spring supported column with the intermediate spring support stiffness.

3, while for higher values of $\zeta$ (such as $\zeta = 0.8$, 0.803 and 1.0), there exists a transition value of $\beta_{Tl}$ where the critical buckling load jumps and the type of instability mechanism is changed. As $\beta_{Tl}$ is less than this transition value, the type of instability mechanism is flutter, while if $\beta_{Tl}$ is greater than this transition value, the type of instability mechanism is divergence. As $\zeta < 0.803$, the critical buckling load increases through the jump as $\beta_{Tl}$ is increased, whereas for $\zeta \geq 0.803$ the critical buckling load decreases through the jump as $\beta_{Tl}$ is increased. This also indicates that with the increase in the value of $\zeta$, the jump occurs at a lower value of $\beta_{Tl}$ and after the jump, the critical buckling load decreases monotonically to a fixed value as $\beta_{Tl}$ is increased. The results also show that when $\beta_{Tl}$ is small, the variation of $\zeta$ has no significant influence on the critical buckling load of flutter instability.

If one considers a column with the boundary conditions as shown in Fig. 5, then when the left-end rotational spring stiffness ($\beta_{\theta L}$) becomes zero, it yields to a hinged-intermediate spring supported column. Instead, if the rotational spring stiffness approaches infinity, it yields to a clamped-intermediate spring supported column. It can be observed from Fig. 5 that when $\zeta$ is small (such as $\zeta = 0.1$), the instability mechanism is flutter, which is due to the coalescence of the first and second natural frequencies, and the critical buckling load increases with increasing $\beta_{\theta L}$. For intermediate values of $\zeta$ (such as $\zeta = 0.5$ and 0.6), the critical buckling load increases with increasing $\beta_{\theta L}$ and the instability mechanism changes from divergence to flutter at a transition point of $\beta_{\theta L}$ where the flutter instability is due to the coalescence of the second and third natural frequencies, and there exists no jump of the critical buckling load. However, there exists a jump of the critical buckling load of flutter instability for the curve $\zeta = 0.5$ at $\beta_{\theta L} = 11.1$; this is due to the flutter instability being changed from the coalescence of the second and third natural frequencies to that of the first and second natural frequencies. After the jump, the critical buckling load decreases with increasing $\beta_{\theta L}$. For higher values of $\zeta$ (such as $\zeta = 1.0$), the instability mechanism becomes divergence for all values of $\beta_{\theta L}$ and the critical buckling load increases monotonically to a fixed value with increasing $\beta_{\theta L}$.

If one considers a column with the boundary conditions as shown in Fig. 6, then when the left-end translational spring stiffness ($\beta_{TL}$) becomes zero, it yields to a guided-intermediate spring supported column. Alternatively, if the translational spring stiffness approaches infinity, it yields to a clamped-intermediate spring supported column. The conclusions are similar to those for the beam shown in Fig. 5, except that when the translational spring stiffness is small, the critical buckling load of divergence instability for higher values of $\zeta$ (such as 0.5, 0.6 and 1.0) is less than that of flutter instability for lower values of $\zeta$ (such as $\zeta = 0.1$).

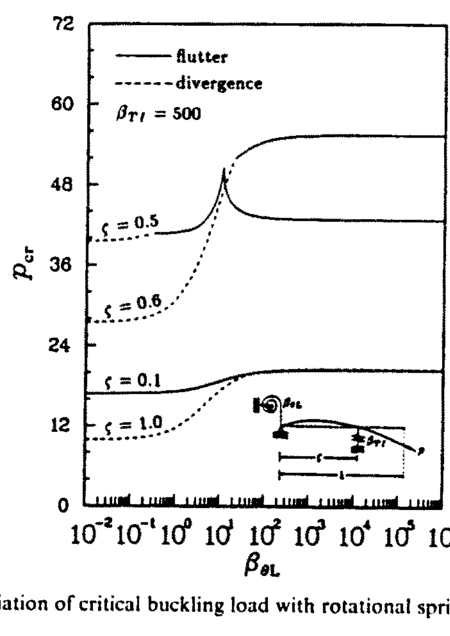


Fig. 5. Variation of critical buckling load with rotational spring stiffness.

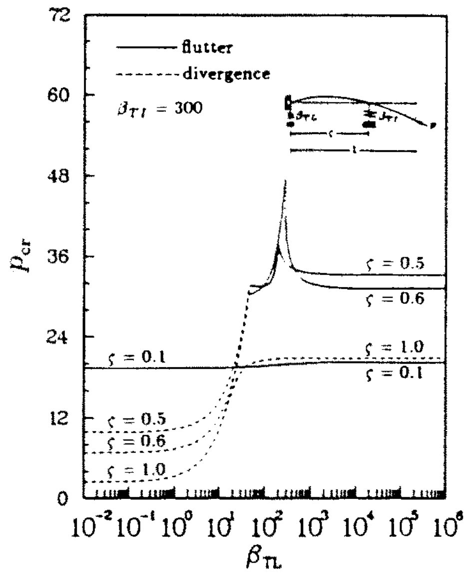


Fig. 6. Variation of critical buckling load with translational spring stiffness.

## 4. CONCLUSIONS

In this paper, the non-conservative stability of an intermediate spring supported uniform column elastically restrained at one end and subjected to a follower force at the other unsupported end is studied. It is found that when the intermediate spring support is far from the unsupported end, the instability mechanism is that of flutter. As the intermediate spring support approaches the unsupported end, the instability mechanism changes from flutter to divergence with increasing intermediate spring stiffness. For the hinged-intermediate and guided-intermediate spring supported columns, the critical buckling load of flutter instability will first decrease, then increase as the intermediate spring stiffness is increased. Nevertheless, when the instability mechanism is divergence, the critical buckling load depends on the location of the intermediate spring support only, whereas for the clamped-intermediate spring supported column the critical buckling load of divergence instability decreases monotonically to a fixed value as the intermediate spring stiffness is increased. For a hinged-intermediate spring supported column with a rotational spring at the hinged end, as the intermediate spring support is far from the unsupported end, the critical buckling load of flutter instability increases with increasing rotational spring stiffness. When the intermediate spring support approaches the unsupported end, the instability mechanism changes from divergence to flutter as the rotational spring stiffness increases. The influence of translational spring stiffness on the critical buckling load of the guided-intermediate spring support with a translational spring at the guided end is similar to those for the hinged-intermediate spring supported column with a rotational spring at the hinged end.

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## APPENDIX: ELEMENTS OF THE MATRIX $[D_{ij}]$

$$D_{11} = S_1 \cosh \lambda\zeta - S_2 \sinh \lambda\zeta + \cos \eta\zeta,$$

$$D_{12} = S_3 \cosh \lambda\zeta - S_4 \sinh \lambda\zeta + \sin \eta\zeta,$$

$$D_{13} = -(S_5 \cosh \lambda\zeta + S_6 \sinh \lambda\zeta + \cos \eta\zeta),$$

$$D_{14} = -(S_7 \cosh \lambda\zeta + S_8 \sinh \lambda\zeta + \sin \eta\zeta),$$

$$D_{21} = S_1\lambda \sinh \lambda\zeta - S_2\lambda \cosh \lambda\zeta - \eta \sin \eta\zeta,$$

$$D_{22} = S_3\lambda \sinh \lambda\zeta - S_4\lambda \cosh \lambda\zeta + \eta \cos \eta\zeta,$$

$$D_{23} = -(S_5\lambda \sinh \lambda\zeta + S_6\lambda \cosh \lambda\zeta - \eta \sin \eta\zeta),$$

$$D_{24} = -(S_7\lambda \sinh \lambda\zeta + S_8\lambda \cosh \lambda\zeta + \eta \cos \eta\zeta),$$

$$D_{31} = S_1\lambda^2 \cosh \lambda\zeta - S_2\lambda^2 \sinh \lambda\zeta - \eta^2 \cos \eta\zeta,$$

$$D_{32} = S_3\lambda^2 \cosh \lambda\zeta - S_4\lambda^2 \sinh \lambda\zeta - \eta^2 \sin \eta\zeta,$$

$$D_{33} = -(S_5\lambda^2 \cosh \lambda\zeta + S_6\lambda^2 \sinh \lambda\zeta - \eta^2 \cos \eta\zeta),$$

$$D_{34} = -(S_7\lambda^2 \cosh \lambda\zeta + S_8\lambda^2 \sinh \lambda\zeta - \eta^2 \sin \eta\zeta),$$

$$D_{41} = S_1\lambda^3 \sinh \lambda\zeta - S_2\lambda^3 \cosh \lambda\zeta + \eta^3 \sin \eta\zeta,$$

$$D_{42} = S_3\lambda^3 \sinh \lambda\zeta - S_4\lambda^3 \cosh \lambda\zeta - \eta^3 \cos \eta\zeta,$$

$$D_{43} = -[S_5(\lambda^3 \sinh \lambda\zeta + \beta_{rl} \cosh \lambda\eta) + S_6(\lambda^3 \cosh \lambda\zeta + \beta_{rl} \sinh \lambda\eta) + (\beta_{rl} \cos \eta\zeta + \eta^3 \sin \eta\zeta)],$$

$$D_{44} = -[S_7(\lambda^3 \sinh \lambda\zeta + \beta_{rl} \cosh \lambda\eta) + S_8(\lambda^3 \cosh \lambda\zeta + \beta_{rl} \sinh \lambda\eta) + (\beta_{rl} \sin \eta\zeta - \eta^3 \cos \eta\zeta)],$$

where

$$S_1 = \frac{\lambda^2\eta^2 - \beta_{0L}\beta_{rL}}{\lambda^4 + \beta_{0L}\beta_{rL}},$$

$$S_2 = \frac{(\lambda^2 + \eta^2)\beta_{rL}}{\lambda^3 + \lambda\beta_{0L}\beta_{rL}},$$

$$S_3 = \frac{(\lambda^2\eta + \eta^3)\beta_{0L}}{\lambda^4 + \beta_{0L}\beta_{rL}},$$

$$S_4 = \frac{\eta\beta_{0L}\beta_{rL} - \lambda^2\eta^3}{\lambda^3 + \lambda\beta_{0L}\beta_{rL}},$$

$$S_5 = \frac{1}{\lambda^5}(\lambda^3\eta^2 \cosh \lambda \cos \eta + \lambda^2\eta^3 \sinh \lambda \sin \eta),$$

$$S_6 = \frac{1}{\lambda^5}(-\lambda^3\eta^2 \sinh \lambda \cos \eta - \lambda^2\eta^3 \cosh \lambda \sin \eta),$$

$$S_7 = \frac{1}{\lambda^5}(\lambda^3\eta^2 \cosh \lambda \sin \eta - \lambda^2\eta^3 \sinh \lambda \cos \eta),$$

$$S_8 = \frac{1}{\lambda^5}(-\lambda^3\eta^2 \sinh \lambda \sin \eta + \lambda^2\eta^3 \cosh \lambda \cos \eta).$$